

SECTION C — ($3 \times 10 = 30$ marks)

Answer any THREE questions.

16. Prove that L is algebraic extension of F , where L is algebraic extension of K and K is algebraic extension of F .
17. Show that any splitting fields E and E^1 of the polynomials $f(x) \in F[x]$ and $f^1(t) \in F^1[t]$ respectively are isomorphic by an isomorphism ϕ with a property $\phi(\alpha) = \alpha^1, \forall \alpha \in F$.
18. Prove that $G(K, F)$ is a finite group and $O(G(K, F)) \leq [K, F]$, where K is a finite extension of F .
19. Show that any two finite fields having the same number of elements are isomorphic.
20. Show that the general polynomial of degree $n \geq 5$ is not solvable by radicals.

APRIL/MAY 2025

23PMA21 — ADVANCED ALGEBRA

Time : Three hours

Maximum : 75 marks



SECTION A — ($10 \times 2 = 20$ marks)

Answer ALL questions.

Define algebraic extension with examples.

2. If $a \in k$ is a root of $p(x) \in F[x]$, $F \subset K$, show that $(x - a) \mid p(x)$ in $k[x]$.
3. If F is a field of characteristic $p \neq 0$, prove that the polynomial $x^{p^m} - x \in F[x]$ has distinct roots.
4. Define simple extension of F .
5. Define Galois group.
6. Show that $\mathbb{Q}(\sqrt[3]{2})$ is not a normal extension of $\mathbb{Q}$.

7. Show that the finite field F has p^m elements, show that every $a \in F$ satisfies $a^{p^m} = a$.

8. If the finite field F has p^m elements, prove that $x^{p^m} - x \in F[x]$ factor in $F[x]$ as $x^{p^m} - x = \prod_{\lambda \in F} (x - \lambda)$.

9. Define solvable group.

10. Prove that $N(xy) = N(x)N(y)$, $\forall x, y \in Q$.

SECTION B — (5 × 5 = 25 marks)

Answer ALL questions.

11. (a) If L is a finite extension of F and K is subfield of L which contains F , prove that $[K : F][L : F]$.

Or

(b) Prove that the elements in K which are algebraic over F forms a subfield of K .

12. (a) Show that there is an extension E of F of degree at most $n!$ in which $f(x)$ has n roots, $f(x) \in F[x]$.

Or

(b) If $\sigma : F \rightarrow F^1$ is an isomorphism, prove that there is an isomorphism $\sigma^* : F[x] \rightarrow F^1[t]$.

13. (a) Show that fixed field of G is a subfield of k .

Or

(b) Prove that C is a normal extension of R .

14. (a) Prove that for every prime number p and every positive integer m there is a unique field having p^m elements.

Or

(b) Prove that a finite field F has p^m elements where p is prime characteristic of F .

15. (a) Show that if $a \in H$ then $a^{-1} \in H$ if and only if $N(a) = 1$.

Or

(b) Suppose F has all n^{th} roots of unity and $a \neq 0 \in F$. Let $x^n - a \in F[x]$ and let k be its splitting field over F . Prove that $K = F(u)$, u is root of $x^n - a$.

